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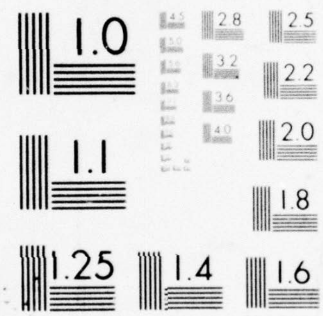


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# ON THE POLYHEDRALITY OF THE CONVEX HULL OF THE FEASIBLE SET OF AN INTEGER PROGRAM

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# ON THE POLYHEDRALITY OF THE CONVEX HULL OF THE FEASIBLE SET OF AN INTEGER PROGRAM

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## ABSTRACT

Polyhedrality is established for convex hulls of sets defined by systems of equations in non-negative integer variables. This property is useful for certain existence, duality, and sensitivity results in integer programming. The structural theorems obtained also shed some light on the relationship between the convex hull and the relaxation obtained by deleting integrality constraints.

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# ON THE POLYHEDRALITY OF THE CONVEX HULL OF THE FEASIBLE SET OF AN INTEGER PROGRAM

R. R. Meyer and M. L. Wage†

## 1. Introduction

A number of results dealing with existence [9] duality [1], and sensitivity analysis for integer programming have been established for integer programs whose feasible sets have convex hulls that are polyhedral (i.e., the intersection of a finite number of closed half-spaces). This is because, given a set $S \subseteq \mathbb{R}^n$ and a linear function $cx$, if the convex hull of $S$ (denoted conv S) is polyhedral, then the problem $\sup cx$ s.t. $x \in S$ has the same optimal value as the linear program $\max cx$ s.t. $x \in \operatorname{conv} S$ (including the infeasible case in which the optimal value is set to $-\infty$, and the unbounded case in which the optimal value is taken as $+\infty$), and, moreover, every optimal extreme point of the linear program is an optimal solution of the problem over $S$. In this report, polyhedrality is established for the convex hull of an arbitrary set $S$ of the form

$$(1) \qquad S \equiv \{x \mid Ax = b,\ x \geq 0,\ x \text{ integer}^*\}$$

$$\text{where} \qquad x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n ,$$

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* A vector or matrix is termed integer or rational if all its elements are respectively integer or rational.

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$A$ is a given $m \times n$ matrix of real numbers, and $b$ is a given element of $\mathbb{R}^m$. While it might be thought that the polyhedrality of the convex hull of the feasible set of an integer program could be taken for granted, it has been shown that in the inequality constrained case, the convex hull may be quite complex [4], [5], [13] and, in fact, need not be polyhedral [3]. In the case of rational coefficients (for both inequalities and equations), polyhedrality was previously proved in [9]. Here we will show that this rationality hypothesis is not required in the equality-constrained case.

## 2. A Rational Representation

In this section it will be shown that the set $S$ defined by (1) always has an equivalent representation as $\{x \mid A'x = b', \ x \geq 0, \ x \text{ integer}\}$, where $A'$ and $b'$ are rational. Once this result is established, polyhedrality of conv $S$ can be demonstrated via Theorem 3.9 of [9]. (However, a more compact and geometrically-motivated proof is possible due to absence of the continuous variables allowed in [9], and this alternative method of proof is given in Section 3.)

Theorem 1 employs the concept of rational independence: a set of real numbers $\{\gamma_1, \ldots, \gamma_k\}$ is said to be rationally independent if $\gamma_1 r_1 + \ldots + \gamma_k r_k = 0$, where $r_1, \ldots, r_k$ are rational, implies $r_1 = \ldots = r_k = 0$. Rational independence of a set of n-vectors is similarly defined. (It is easily seen that rational independence and integral independence, i.e., independence with respect to integer weights, are equivalent, but due to the mechanics of the proofs to follow, rational independence is more convenient to work with. Note that while linear independence clearly implies rational independence, the converse is not true.)

<u>Theorem 1:</u> Let $S_e = \{x \mid Ax = b,\ x \text{ integer}\}$. Then there exists an $m' \times n$ matrix $A'$ of rationals and a vector $b'$ of rationals such that $S_e = \{A'x = b',\ x \text{ integer}\}$.

<u>Proof:</u> If $S_e = \emptyset$, then we may take $m' = 1$, $A = 0$, $b = 1$ and achieve the required rational representation, so we may assume $S_e \neq \emptyset$. We will first consider the case in which the system $Ax = b$ consists of a <u>single</u> equation, since a similar analysis performed <u>equation-by-equation</u> will yield the desired result in the general case. Denote the single equation by

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = \beta \ . \tag{2}$$

If all $\alpha_i = 0$, then feasibility implies $\beta = 0$, so no transformation is needed. Thus, we may assume that not all $\alpha_i$ are 0, and, for notational convenience, we also assume that the variables have been ordered so that $\alpha_1 \neq 0$ (in dealing with a system of equations, a different ordering might be required for each equation, but this causes no problems). If $n = 1$ then $\beta = \alpha_1 x_1^*$ for some integer $x_1^*$, and the data may be rationalized by dividing through by $\alpha_1$.

If $n > 1$, we replace coefficients by rational combinations of "previous", "independent" coefficients whenever possible. Thus, if $\alpha_2 = \alpha_1 r_{1,2}$, where $r_{1,2}$ is rational, we re-write (2) in the form

(3) $$\alpha_1(x_1 + r_{1,2}x_2) + \ldots + \alpha_n x_n = \beta .$$

Continuing this procedure, we end up with an index set $I \subseteq \{1, \ldots, n\}$ such that (2) is equivalent to

(4) $$\sum_{i \in I} (\alpha_i \sum_{j=1}^{n} r_{i,j} x_j) = \beta ,$$

where the $r_{i,j}$ are rational, and the $\alpha_i$ for $i \in I$ are rationally independent. Since $S_e \neq \emptyset$, $\sum_{i=1}^{n} \alpha_i x_i^* = \beta$ for some integers $x_1^*, \ldots, x_n^*$, and by carrying out the same conversion as above we have

(5) $$\sum_{i \in I} (\alpha_i \sum_{j=1}^{n} r_{i,j} x_j^*) = \beta ,$$

By subtracting (5) from (4) we have

(6) $$\sum_{i \in I} (\alpha_i \sum_{j=1}^{n} r_{i,j}(x_j - x_j^*)) = 0 .$$

We will now show that

(7) $$S_e = \{x \mid \sum_{j=1}^{n} r_{i,j} x_j = \sum_{j=1}^{n} r_{i,j} x_j^* \text{ for } i \in I, \ x \text{ integer}\} .$$

Clearly, if $x$ is integer and satisfies the equations of (7), then $x \in S_e$, so suppose that $x \in S_e$, but that $\sum_{j=1}^{n} r_{i,j} x_j \neq \sum_{j=1}^{n} r_{i,j} x_j^*$ for at least one $i$.

From (6) it would follow that the $\alpha_i$ with $i \in I$ were not rationally independent, which is a contradiction of the way in which they were constructed. Thus in the single equation case, the set $S_e$ has an equivalent representation of the form (7).

When $Ax = b$ consists of more than one equation, an analogous procedure may be performed for each equation in the system, so that $S_e$ may be represented in terms of the collection of the corresponding systems of the form (7). Alternatively, we may express $A$ and $b$ in terms of a "basis" of rationally independent columns of $A$, and carry out a proof analogous to that of the scalar case. ∎

Example 1: Let $S_e \equiv \{x \mid -1 \cdot x_1 - \frac{1}{2}\sqrt{2}\, x_2 + \frac{4}{3} \cdot x_3 = \frac{5}{3} - \sqrt{2},\ x \text{ integer}\}$. $-1$ and $-\frac{1}{2}\sqrt{2}$ are rationally independent, but $4/3 = -1 \cdot -4/3 - \frac{1}{2}\sqrt{2} \cdot 0$, so the equation in $S_e$ may be written as $-1 \cdot (x_1 - \frac{4}{3}x_3) - \frac{1}{2}\sqrt{2} \cdot (x_2) = \frac{5}{3} - \sqrt{2}$. Since setting $x_1 = 1$, $x_2 = 2$, $x_3 = 2$ yields a point in $S_e$, we may write $5/3 - \sqrt{2} = -1 \cdot (1) - \frac{1}{2}\sqrt{2} \cdot (2) + \frac{4}{3} \cdot (2)$. Substituting for the original RHS, the equation in $S_e$ is written as $-1 \cdot (x_1 - \frac{4}{3}x_3) - \frac{1}{2}\sqrt{2} \cdot (x_2) = -1 \cdot (1) - \frac{1}{2}\sqrt{2} \cdot (2) + \frac{4}{3} \cdot (2) = -1 \cdot (-\frac{5}{3}) - \frac{1}{2}\sqrt{2} \cdot (2)$. From the preceding theorem we conclude that $S_e$ may be written as $\{x \mid x_1 - \frac{4}{3}x_3 = -\frac{5}{3},\ x_2 = 2,\ x \text{ integer}\}$.

Example 2:

$$x_1 + \sqrt{2}\, x_2 + \frac{1}{2} x_3 = \frac{5}{2} + 2 \cdot \sqrt{2}$$

$$-x_1 + x_2 + x_3 - 3x_4 = -2$$

$$x_i \text{ integer } (i = 1, \ldots, 4) .$$

Using the equation-by-equation approach, we find that only the first equation needs to be "rationalized", and it yields the two equations $x_1 + \frac{1}{2}x_3 = \frac{5}{2}$ and $x_2 = 2$, so that the original system is equivalent to

$$x_1 + \frac{1}{2} x_3 = \frac{5}{2}$$

$$x_2 = 2$$

$$-x_1 + x_2 + x_3 - 3 x_4 = -2$$

$$x_i \text{ integer } (i = 1, \ldots, 4) .$$

On the other hand, considering the rational independence of the columns, we find that the first three columns of A are rationally independent, whereas the fourth column can be expressed as a rational combination (with weights 1 and -2) of columns 1 and 3. Thus, the original system may also be shown to be equivalent to:

$$x_1 + x_4 = 3$$

$$x_2 = 2$$

$$x_3 - 2x_4 = -1$$

$$x_i \text{ integer } (i = 1, \ldots, 4) . \quad \blacksquare$$

Note that if there exists a number $\gamma$ such that all of the quotients $\alpha_1/\gamma, \ldots, \alpha_n/\gamma, \beta/\gamma$ are rational, then (assuming $\alpha_1 \neq 0$) $(\alpha_j/\gamma)/(\alpha_1/\gamma) = \alpha_j/\alpha_1$ is rational for $j = 2, \ldots, n$, so the conversion procedure in Theorem 1 yields $I = \{1\}$, and, in fact, that procedure is simply equivalent to dividing through by $\gamma$. (In fact, if there exists a $\gamma$ such that the quotients $\alpha_1/\gamma, \ldots, \alpha_n/\gamma$ are all rational, then there exists a $\gamma'$ such that $\alpha_1/\gamma', \ldots, \alpha_n/\gamma'$ are all integer, and in this case integrality of $\beta/\gamma'$ is clearly a necessary condition for the existence of an integer solution. Under this divisibility assumption, a necessary and sufficient condition for the existence of an integer solution is that the "generalized greatest common divisor" (see [10]) of $\alpha_1, \ldots, \alpha_n$ "divide" $\beta$ in the sense of giving an integer quotient.) However, as the preceding numerical example shows, the coefficients need not have this divisibility property, and in such a case $I \neq \{1\}$ and a single equation will be converted into an equivalent system of equations.

Corollary 1 below gives two additional results easily obtained from Theorem 1 and an analysis of its proof.

Corollary 1: If $S_e = \{x \mid Ax = b, \ x \text{ integer}\}$ and $S_e^R = \{x \mid Ax = b, \ x \text{ rational}\}$, then there exist integer $\hat{A}$ and $\hat{b}$ such that $S_e = \{x \mid \hat{A}x = \hat{b}, \ x \text{ integer}\}$ and $S_e^R = \{x \mid \hat{A}x = \hat{b}, \ x \text{ rational}\}$.

Proof: Using Theorem 1, $S_e$ may be written as $\{x \mid A'x = b', \ x \text{ integer}\}$, where $A'$ and $b'$ are rational, and by multiplying each equation of $A'x = b'$ by a suitable integer, conversion to integer data is achieved. Inspection of the proof of Theorem 1 shows that all of the steps go through if the $x_i$ are assumed rational rather than integer. ∎

An alternative approach to the derivation of the results of this section is to consider the set $S_e^R$, and, by using the fact that a linear transformation between two vector spaces over a field (in this case, the rationals) has a representation in terms of a matrix whose elements come from the field, show that $S_e^R$ has a rational representation. Theorem 1 then follows by taking the intersection of $S_e^R$ with the integer vectors in $\mathbb{R}^n$.

In Section 3 the rational representation of S is used to prove polyhedrality of conv S, however, it might also be noted that other useful structural properties (see [6]) can also be derived from the rational representation.

## 3. Structural Properties

In establishing the polyhedrality of conv S, we may assume by Theorem 1 that S is represented in the form

$$S = \{x \mid A'x = b', \quad x \geq 0, \quad x \text{ integer}\} , \tag{8}$$

where A' and b' are rational.

In order to state structural properties of conv S in a compact form, we introduce the following definitions:

$$E \equiv \{x \mid x \text{ is an extreme point of } S\}$$

$$K' \equiv \{x \mid A'x = 0, \quad x \geq 0\}$$

$$K'_R \equiv \{x \mid x \in K', \ x \text{ rational}\}$$

$$K'_T \equiv \{x \mid x \in K', \ x \text{ integer}\}$$

$$Z^n \equiv \{x \mid x \in \mathbb{R}^n, \ x \text{ integer}\} .$$

The following Lemma 1 leads immediately to the finiteness of E. For completeness, a proof of Lemma 1 is given, although an equivalent result stated in a slightly different manner is given as Lemma 4.1 of [7], where it is used as the basis for a proof of the "König Infinity Lemma" [8].

Lemma 1 (Dominance Lemma):

Let $\{p^{(1)}, p^{(2)}, \ldots\}$ be an infinite sequence of distinct, non-negative points of $Z^n$. Then there exist distinct indices $i, j$ such that $p^{(i)} \leq p^{(j)}$ (componentwise).

Proof. We first show that there exist index sets $J'$ and $J''$ (with $J' \cup J'' = \{1, \ldots, n\}$) and an increasing subsequence $I_n'$ of the integers such that the sequences $\{p_i^{(k)}\}$ are constant for $i \in J'$ and have certain useful properties for $i \in J''$. Consider the sequence $\{p_1^{(k)}\}$, and, if for each $N$ there exists a $k(N)$ such that $k \geq k(N)$ implies $p_1^{(k)} \geq N$, set $I_1' \equiv \{1, 2, 3, \ldots\}$ and put 1 in the index set $J''$. Otherwise, for some $N$, there exists an increasing subsequence of integers $I_1$ such that $k \in I_1$ implies $p_1^{(k)} < N$, and thus there exists a subsequence $I_1'$ of $I_1$ such that $p_1^{(k)} = \bar{p}_1$, a constant, for all $k \in I_1'$; put 1 in the index set $J'$. Now carry out the analogous procedure for the sequence $\{p_2^{(k)} | k \in I_1'\}$, thereby obtaining a subsequence $I_2'$ of $I_1'$ and assigning 2 to $J'$ or $J''$. In general, given the index set $I_i'$ we similarly construct $I_{i+1}'$ and place $i + 1$ in $J'$ or $J''$ $(i = 1, \ldots, n-1)$. Note that those indices $i \in J'$ have the property stated above, and those indices $i \in J''$ have the property that, for every $N$, there exists a $k(N)$ such that $k \geq k(N)$ implies $p_i^{(k)} \geq N$. Choosing an arbitrary $i \in I_n'$, it is clear that by choosing a sufficiently large $j \in I_n'$, the inequality $p^{(i)} \leq p^{(j)}$ will hold. ∎

**Lemma 2:** $|E|$ is finite.

**Proof:** If the result were false, by the Dominance Lemma, there would be two extreme points $x'$ and $x''$ satisfying $x' \leq x''$ with $x' \neq x''$. However, $x' + 2(x'' - x')$ is easily seen to be in $S$, and the equation $x'' = \frac{1}{2} x' + \frac{1}{2} [x' + 2(x'' - x')]$ contradicts the hypothesis that $x''$ was an extreme point of $S$. ■

(Lemma 2 was proved in [9], but the above proof is more compact and offers more geometric insight.)

**Theorem 2:** $S = (\text{conv } E + K'_R) \cap Z^n$.

**Proof:** In Theorem 3.6 of [9], it was shown that every point of $S$ is contained in $\text{conv } E + K'_R$. Conversely, by verifying that the constraints are satisfied, it is easily seen that every integer point of $\text{conv } E + K'_R$ is in $S$. ■

**Lemma 3:** $K' = \text{conv } K'_R = \text{conv } K'_I$.

**Proof:** Since $K'$ is convex and $K' \supseteq K'_R \supseteq K'_I$, $K' \supseteq \text{conv } K'_R \supseteq \text{conv } K'_I$. The proof will be completed by showing that $\text{conv } K'_I \supseteq K'$. Since $K'$ is a polyhedral cone, if $x \in K'$, then $x = \sum_{j=0}^{\omega} \mu_j r_j$, where the $\mu_j$ are non-negative weights and the $r_j$ may be taken as the extreme points of $K' \cap \{x | \sum_{i=1}^{n} x_i \leq 1\}$. (We assume $r_0 = 0$.) Since $A'$ is rational, however, these extreme points are easily seen to be rational. Let $N$ be any positive integer such that (1) $Nr_j$ is integer for all $j$ and (2) $N \geq \sum_{j=0}^{\omega} \mu_j$ so that $x = \sum \bar{\mu}_j \bar{r}_j$, where $\bar{\mu}_j = \mu_j/N$ and $\bar{r}_j = Nr_j$. Since $\bar{r}_0 = 0$, we may write $x = [\bar{\mu}_0 + (1 - \sum_{j=0}^{\omega} \bar{\mu}_j)]\bar{r}_0 + \sum_{j=1}^{\omega} \bar{\mu}_j \bar{r}_j$, so that $x$ has been expressed as a convex combination of $\bar{r}_j \in K'_I$. ■

<u>Theorem 3:</u> conv S = conv E + K'.

<u>Proof:</u> From Theorem 2 it follows that $S \subseteq \text{conv } E + K'_R \subseteq \text{conv } E + K'$. Since conv E + K' is convex, conv S $\subseteq$ conv E + K'. To get the opposite inequality, note that by Lemma 3 conv E + K' = conv E + conv $K'_I$, so that conv E + K' = conv (E + $K'_I$) $\subseteq$ conv S, since (E + $K'_I$) $\subseteq$ S. ■

Since the sets conv E and K' are polyhedral, Theorem 3 establishes the <u>polyhedrality</u> of conv S.

Theorem 3 also allows an interesting comparison to be made between conv S and the <u>linear programming relaxation</u> T' of S defined by $T' \equiv \{x \mid A'x = b', \ x \geq 0\}$. Since T' is polyhedral and line-free, we have (see [ ]) T' = conv E' + K', where E' is the set of extreme points of T'. Comparing this with Theorem 2, we note that, roughly speaking, conv S and T' "coincide" in their asymptotic parts and "differ" only in their extreme points. (From a computational point of view, however, this difference is crucial, since the extreme points of T' have a nice algebraic characterization (as basic feasible solutions) and have a cardinality that is bounded from above by $\binom{n}{m'}$, whereas these properties do not carry over to the extreme points of S.) This property is <u>not</u> the case, however, for the linear programming relaxation $T$ defined in terms of the original constraints by $T \equiv \{x \mid Ax = b, \ x \geq 0\}$, as may be seen by considering the following example: if the constraints $Ax = b$ are given by $x_1 - \sqrt{2}\, x_2 = 1 - \sqrt{2}$, then $T$ consists of the ray $\{x \mid x_1 - \sqrt{2}\, x_2 = 0, \ x_1 \geq 0, \ x_2 \geq 0\}$, whereas $T' = \{x \mid x_1 = 1, \ x_2 = 1\} = \{(1, 1)\}$. Of course if $A$ is rational, then $T$ and $T'$ coincide, but without hypotheses on $A$, it is only possible to conclude that $T \supseteq T'$ and that $K \equiv \{x \mid Ax = 0, \ x \geq 0\} \supseteq K'$ (that $A'x = b'$ implies $Ax = b$ and that $A'x = 0$ implies $Ax = 0$ are easily seen from (7)). These results are summarized in the following theorem, where $E^*$ denotes the set of extreme points of $T$.

Theorem 4: $\text{conv } S = \text{conv } E + K' \subseteq \text{conv } E' + K' = T' \subseteq T = \text{conv } E^* + K$ and $K' \subseteq K$.

Proof: Since $S \subseteq T'$ and $T'$ is convex, $\text{conv } S \subseteq T'$. The other relations have been previously discussed. ∎

Note also that Theorem 4 implies that if $T$ is a bounded set, then $K' = K = \{0\}$ and $\text{conv } E \subseteq \text{conv } E' \subseteq \text{conv } E^*$. However, if $T$ and $T'$ are unbounded, then no ordering relations need hold between $E$, $E'$, or $E^*$ (or between their convex hulls), as may be seen by considering Example 1, in which the corresponding sets are $E = \{(1, 2, 2)\}$, $E' = \{(0, \frac{5}{4}, 2)\}$, and $E^* = \{(0, 0, \theta)\}$, where $\theta = (\frac{5}{3} - \sqrt{2})/(\frac{4}{3})$.

Polyhedrality of $\text{conv } S$ may also be demonstrated directly without resorting to the rational representation of Theorem 1. Defining $K_R \equiv \{x \mid Ax = 0, \ x \text{ rational}\}$, we may prove along the lines of the proof of

Theorem 3 that conv $S$ = conv $E$ + conv $K_R$, so that polyhedrality of conv $S$ will follow from the polyhedrality of conv $K_R$. Polyhedrality of conv $K_R$ is established by considering span $K_R$, i.e. the set of all linear combinations of elements of $K_R$, and using the following lemma:

Lemma 4: If $x \in$ span $K_R \cap \mathbb{R}^n_+$, then there is a rational $\bar{x} \in$ span $K_R \cap \mathbb{R}^n_+$ with $\bar{x}_i = 0$ if and only if $x_i = 0$ $(i = 1, \ldots, n)$.

Proof: First note that since span $K_R$ is the span of rational vectors, any maximal independent subset of $K_R$ forms a rational basis for span $K_R$. Fix such a maximal independent set and let $B$ be the matrix whose $i^{th}$ column is the $i^{th}$ vector in this independent set. Then

$$\text{span } K_R = \{B\alpha \mid \alpha \in \mathbb{R}^K\} .$$

Fix $x \in$ span $K_R \cap \mathbb{R}^n_+$. Then there exists an $\alpha$ such that $B\alpha = x$. Let $\hat{B}$ be the matrix consisting of those rows, $b_i$, of $B$ for which $x_i = 0$. Since $\hat{B}$ is rational, the null space of $\hat{B}$ has a basis consisting of the columns of a rational matrix $C$. Hence $\alpha = C\gamma$ for some $\gamma$. Perturb $\gamma$ slightly to get a rational $\gamma'$. Then $\alpha' = C\gamma'$ is rational, $\hat{B}\alpha' = 0$, and with a small enough perturbation $\alpha'$ is sufficiently close to $\alpha$ so that $B\alpha'$ has positive components where $B\alpha$ has positive components. Let $\bar{x} = B\alpha'$. ∎

Theorem 5: conv $K_R$ = span $K_R \cap \mathbb{R}^n_+$.

Proof: Since span $K_R \cap \mathbb{R}^n_+$ is convex, it suffices to show that conv $K_R \supseteq$ span $K_R \cap \mathbb{R}^n_+$. Fix $x \in$ span $K_R \cap \mathbb{R}^n_+$. We simply drive each coordinate of $x$ to zero by subtracting appropriate multiples of rationals in span $K_R \cap \mathbb{R}^n_+$.

Specifically, using Lemma 4, choose a rational $r_1 \in$ span $K_R \cap \mathbb{R}^n_+$ that has the same zero coordinates as $x$. Then there exists a number, $\gamma_1$, such that $x - \gamma_1 r_1$ is non-negative and has more zero coordinates than $x$. Continue choosing $r$'s and $\gamma$'s so that at each step, $j$, $x - \sum_{i<j} \gamma_i r_i$ is non-negative and has more zero coordinates than $x - \sum_{i<j-1} \gamma_i r_i$. This process must stop at some $j_0 \leq n$ with $x - \sum_{i<j_0} \gamma_i r_i = 0$, i.e. $x = \sum_{i<j_0} \gamma_i r_i$, where the weights $\gamma_i$ are non-negative. By adjusting the weights and $r_i$ as in the proof of Lemma 3, it may be shown that $x \in$ conv $K_R$. ■

In closing, it should be re-iterated that in the inequality *constrained* case, if we define $S_I = \{x | Ax \leq b, \ x \geq 0, \ x \text{ integer}\}$, it need not be the case that conv $S_I$ is polyhedral. This may be seen from the following problem considered in [9]:

$$\begin{aligned} &\text{maximize } -\alpha x_1 + x_2 \\ &\text{s.t.} \quad -\alpha x_1 + x_2 \leq 0 \\ &\qquad\qquad x_1 \geq 1 \\ &\qquad\qquad x_2 \geq 0 \\ &\qquad\qquad x_1, x_2 \text{ integer}. \end{aligned} \tag{9}$$

It was shown that the problem (9) does not have an optimal solution if $\alpha$ is any positive rational, even though it is feasible and not unbounded. This phenomenon could not occur if the convex hull of the feasible set were polyhedral, since that property would guarantee the existence of an optimal solution. (In this particular example, it may be shown that the corresponding $S_I$ actually has an infinite number of extreme points (see [3] for related work). Moreover, by replacing $\alpha$ by rationals suitably close to $\alpha$, and by replacing the variable $x_1$ by $x_1' = x_1 - 1$, it may be shown that, in the equality-constrained case, the number of extreme points of S can be made arbitrarily large if $n \geq 3$, i.e., if the number of variables is at least 3. This contrasts with the equality-constrained cases in which $n = 1$ and 2, where from the geometry of S it is clear that maximum number of extreme points is 1 and 2 respectively. For related complexity results in the inequality-constrained case see [3], [4], [5], [13].)

However, if the matrix A is rational, then the constraints of $S_I$ may be converted into an equivalent set of equations in integer variables, so that the results above may be applied to prove that conv $S_I$ is polyhedral in the rational coefficient case.

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Polyhedrality is established for convex hulls of sets defined by systems of equations in non-negative integer variables. This property is useful for certain existence, duality, and sensitivity results in integer programming. The structural theorems obtained also shed some light on the relationship between the convex hull and the relaxation obtained by deleting integrality constraints.


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